Spin equation and its solutions

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Abstract

The aim of the present article is to study in detail the so-called spin equation (SE) and present both the methods of generating new solution and a new set of exact solutions. We recall that the SE with a real external field can be treated as a reduction of the Pauli equation to the $(0 + 1)$-dimensional case. Two-level systems can be described by an SE with a particular form of the external field. In this article, we also consider associated equations that are equivalent or (in one way or another) related to the SE. We describe the general solution of the SE and solve the inverse problem for this equation. We construct the evolution operator for the SE and consider methods of generating new sets of exact solutions. Finally, we find a new set of exact solutions of the SE.

1   Introduction

We refer as the spin equation (SE) to the following set of two ordinary linear differential equations of first order for the functions $v_1(t)$ and $v_2(t)$:

$$i\dot{V} = (\sigma \mathbf{F}) V , \ V = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} , \ \dot{V} = dV/dt . \tag{1}$$

Here, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, and $\mathbf{F}$ is a given time-dependent (in general, complex) vector,

$$\mathbf{F} = (F_k, \ k = 1, 2, 3) , \ F_k = F_k(t) . \tag{2}$$

In what follows, the column V and the vector $\mathbf{F}$ are called the spinor field and the external field, respectively.

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The SE with a real external field can be treated as a reduction of the Pauli equation [1] to the (0+1)-dimensional case. Such an equation is used to describe a (frozen in space) spin-1/2 particle of magnetic momentum μ, immersed in a magnetic field $\mathbf{B}$ (in this case, $\mathbf{F} = -\mu\mathbf{B}$), and was intensely studied in connection with the problem of magnetic resonance [2, 3]. Besides, complex quantum systems with a discrete energy spectrum can be placed in a special dynamic configuration in which only two stationary states are important. In those cases, it is possible to reduce the Hilbert space of the system to a two-dimensional space. Such two-level systems can also be described by the SE. The SE with an external field of the form $\mathbf{F} = (F_1, 0, F_3)$, $F_1 = \epsilon$, where ϵ is a constant, describes two-level systems with unperturbed energy levels $\pm\epsilon$ ($F_3 \equiv 0$) submitted to an external time-dependent interaction $F_3(t)$, inducing a transition between the unperturbed eigenstates. Two-level systems possess a wide range of applications, for example, the semi-classical theory of laser beams [4], the absorption resonance and nuclear induction experiments [2], the behavior of a molecule in a cavity immersed in electric or magnetic fields [5], and so on. Recently, this subject attracts even more attention due to its relation to quantum computation [6], where the state of each bit of conventional computation is permitted to be any quantum-mechanical state of a *qubit* (quantum bit), which can be treated as a two-level system. The SE with complex external fields describes a possible damping of two-level systems. There exist various equations that are equivalent, or (in a sense) related, to the SE. For example, the well-known top equation, which appears in the gyroscope theory, in the theory of precession of a classical gyromagnet in a magnetic field (see [5]), and so on. The SE with an external field in which $F_s(t)$, $s = 1, 2$ are purely imaginary and F_3 is constant is a degenerate case of the Zakharov–Shabat equation, which plays an important role in the soliton theory [7]. It turns out that two-level systems present a convenient object for illustrating and applying the geometrical phase method [8]. For periodic, or quasiperiodic, external fields, the equations of a two-level system have been studied by many authors using different approximation methods, e.g., perturbative expansions [9], see also [10]. When the external field $\mathbf{F}$ is not periodic, or quasiperiodic, there exists no regular approach to finding exact solutions of the SE. One ought to stress that exact solutions are very important in view of the numerous physical application of the SE. Since in the general case the problem of finding such solutions is sufficiently involved, it has been solved only in a few particular cases.

The first exact solution of the SE was found by Rabi [11], for the external field of the form

$$\mathbf{F} = (f_1 \cos(\Omega t), \ f_2 \sin(\Omega t), \ F_3) \ , \tag{3}$$

where $f_{1,2}$, Ω, and F_3 are some constants. For an external field of the form

$$\mathbf{F} = (F_1, 0, F_3) \ , \ F_1 = \text{const} \ , \tag{4}$$

exact solutions for two different functions F_3 were found in [12]. These functions are

$$F_3 = c_0 \tanh t + c_1 \ , \ F_3 = \frac{c_0}{\cosh t} + c_1 \ , \tag{5}$$

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where $c_{0,1}$ are arbitrary real constants. In the work [14], exact solutions for three sufficiently complicated functions F_3 were found. One (the simplest) of these functions reads

$$F_3 = c_0 + \frac{2\left(c_1^2 - c_0^2\right)}{Q + c_0}, \; Q = \left\{ \begin{array}{l} c_1 \cosh\varphi \, , \; c_1^2 > c_0^2 \\ c_1 \cos\varphi \, , \; c_1^2 < c_0^2 \end{array} \right. ,$$
$$\varphi = 2\left(t\sqrt{|c_1^2 - c_0^2|} + c_2\right) , \tag{6}$$

where $c_{0,1,2}$ are arbitrary real constant.

The aim of the present article is to summarize the properties of the SE, in order to reveal in detail its relation to other types of equations (associated equations), and present both a new set of its exact solutions and methods of generating new solutions.

The article is organized as follows: In sec. 2 we consider associated equations that are equivalent or (in one way or another) related to the SE. In secs. 3 and 4 we describe the general solution of the SE and consider the inverse problem for this equation. A particular, but very important, case of the self-adjoint SE is considered in sec. 5. We describe here its general solution, solve the inverse problem, and derive the associated classical Hamiltonian and Lagrangian systems. In sec. 6 we construct the transformation matrix that relates different solutions of the SE, and construct the evolution operator for the SE. The transformation matrix allows one to generate new sets of exact solution (and the corresponding external fields) from already known ones. In sec. 7, we represent a set of new exact solutions (together with the corresponding external fields) of the SE. These solutions are obtained with the help of a relation (established in sec. 2) between the SE and the one-dimensional Schrödinger equation. In sec. 7, we adapt the Darboux transformation to a particular case of the SE and thus obtain, as an example, a new exact solution. The Appendix contains some technical details related to two-component spinors and associated vectors that are used in the article.

2 Associated equations

Below, we present a set of various equations that are equivalent or (in one way or another) related to the SE. In addition to the definition (1), we note the following:

The equation conjugate to the SE has the form

$$i\dot{V}^+ = -V^+\left(\sigma\mathbf{F}^*\right) , \; V^+ = \left(\begin{array}{cc} v_1^*(t) & v_2^*(t) \end{array} \right) . \tag{7}$$

The inner product of two spinors U and V is defined as

$$(U, V) = U^+V = (u_1^*v_1 + u_2^*v_2) . \tag{8}$$

The SE and its conjugate equation can be written in the component form as follows:

$$i\dot{v}_1 = F_3 v_1 + (F_1 - iF_2)\, v_2\,,\ \ i\dot{v}_2 = -F_3 v_2 + (F_1 + iF_2)\, v_1\,, \tag{9}$$

$$i\dot{v}_1^* = -F_3^* v_1^* - (F_1^* + iF_2^*)\, v_2^*\,,\ \ i\dot{v}_2^* = F_3^* v_2^* - (F_1^* - iF_2^*)\, v_1^*\,. \tag{10}$$

The set (10) can be written in the form of the SE for the spinor $\bar{V}$,

$$\bar{V} = -i\sigma_2 V^* = \begin{pmatrix} -v_2^* \\ v_1^* \end{pmatrix}\,, \tag{11}$$

with an external field $\mathbf{F}^*$,

$$i\dot{\bar{V}} = (\sigma \mathbf{F}^*)\, \bar{V}\,. \tag{12}$$

We refer to the spinor $\bar{V}$ as anti-conjugate to the spinor V.

Sometimes, we represent the external field as

$$\mathbf{F} = \mathbf{K} + i\mathbf{G}\,,\ \ \mathbf{K} = \operatorname{Re}\mathbf{F}\,,\ \ \mathbf{G} = \operatorname{Im}\mathbf{F}\,,$$

$$\mathbf{K} = (K_k)\,,\ \ \mathbf{G} = (G_k)\,,\ \ k = 1, 2, 3\,, \tag{13}$$

where $\mathbf{K}$ and $\mathbf{G}$ are real vectors.

2.1 Associated Schrödinger equations

1. Consider the Schrödinger equation in 0+1 dimensions for a time-dependent two-component complex spinors Ψ. In the general case, the equation has the form

$$i\dot{\Psi} = H\Psi\,, \tag{14}$$

where the Hamiltonian H is a 2×2 complex time-dependent matrix and Ψ a spinor. The matrix H can always be decomposed in the basic matrices, $H = hI + \sigma\mathbf{F}$, where $h = h(t)$ and $\mathbf{F} = (F_k(t)\,,\ k = 1, 2, 3)$. Making the transformation $\Psi = U \exp\left(-i \int h\, dt\right)$, we arrive at the SE for the spinor U.

2. The SE can be reduced to a set of two independent one-dimensional Schrödinger equations with complex potentials in the general case. Consider the set (9) for the function $v_s(t)\,,\ s = 1, 2$. Squaring this set and introducing functions $\psi_s(t)$,

$$v_s = \sqrt{A_s}\psi_s\,,\ \ A_s = F_1 + (-1)^s iF_2\,, \tag{15}$$

we obtain for the latter functions an independent linear differential equation of second order:

$$\ddot{\psi}_s - V_s\psi_s = 0\,,\ \ s = 1, 2\,,$$

$$V_s = \frac{3}{4}\left(\frac{\dot{A}_s}{A_s}\right)^2 - \frac{1}{2}\frac{\ddot{A}_s}{A_s} - A_1 A_2 - F_3^2 - i(-1)^s\left(F_3\frac{\dot{A}_s}{A_s} - \dot{F}_3\right)\,. \tag{16}$$

Each of equations (16) is a one-dimensional Schrödinger equation with the complex potentials V_s.

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2.2 Reduction of the external field

Suppose $V'(t)$ is a solution of the SE with an external field $\mathbf{F}' = (F_1', F_2', F_3')$. Let us perform a transformation

$$V = \hat{T}V', \ \hat{T}(t) = \exp\left[i\alpha\left(\sigma\mathbf{l}\right)\right].\tag{17}$$

Here, $\mathbf{l}$ is an arbitrary constant complex vector, and $\alpha(t)$ is an arbitrary complex function.

If $\mathbf{l}^2 \neq 0$, then, without loss of generality, we can set $\mathbf{l}^2 = 1$, so that the matrix (17) reads

$$\hat{T} = \cos\alpha + i\left(\sigma\mathbf{l}\right)\sin\alpha.$$

One can easily verify that the spinor $V(t)$ also obeys an SE with a reduced external field $\mathbf{F}$ of the form

$$\mathbf{F} = \left[\mathbf{F}' - \mathbf{l}\left(\mathbf{F}'\mathbf{l}\right)\right]\cos\left(2\alpha\right) + \left[\mathbf{F}' \times \mathbf{l}\right]\sin\left(2\alpha\right) + \mathbf{l}\left(\mathbf{F}'\mathbf{l} - \dot{\alpha}\right).\tag{18}$$

Since the transformation matrix $\hat{T}$ is invertible, the SE with the external field $\mathbf{F}'$ and that with the external field $\mathbf{F}$, given by (18), are equivalent. If we subject, for example, the function α to the relation

$$\dot{\alpha} = \mathbf{F}'\mathbf{l} \Longrightarrow \mathbf{F}\mathbf{l} = 0,\tag{19}$$

then the projection of $\mathbf{F}$ onto the direction $\mathbf{l}$ becomes zero, i.e., the reduced external field $\mathbf{F}$ has only two nonzero (complex) components in the plane that is orthogonal to $\mathbf{l}$.

One can also consider complex constant $\mathbf{l}$ with $\mathbf{l}^2 = 0$. Then

$$\hat{T} = 1 + i\alpha\left(\sigma\mathbf{l}\right),$$

and

$$\mathbf{F} = \mathbf{F}' + 2\alpha\left[\mathbf{F}' \times \mathbf{l}\right] + \mathbf{l}\left[2\alpha^2\left(\mathbf{F}'\mathbf{l}\right) - \dot{\alpha}\right].\tag{20}$$

By an appropriate choice of a complex $\mathbf{l}$, one can always reduce to zero any component of both $\mathbf{K} = \operatorname{Re}\mathbf{F}$ and $\mathbf{G} = \operatorname{Im}\mathbf{F}$. However, in this case we cannot imagine $\mathbf{F}$ as a vector in a fixed plane, unlike in the case of a real $\mathbf{l}$.

Let us choose the vector $\mathbf{l}$ to be the unit vector in the z-direction, $\mathbf{l} = (0, 0, 1)$, and α to be a solution of the equation

$$\dot{\alpha} = F_3'.$$

Then the reduced external field $\mathbf{F}$ takes the form

$$\mathbf{F} = (F_1, F_2, 0),\tag{21}$$

where

$$F_1 = F_1'\cos\left(2\alpha\right) + F_2'\sin\left(2\alpha\right), \ F_2 = F_2'\cos\left(2\alpha\right) - F_1'\sin\left(2\alpha\right),$$
$$F_1' = F_1\cos\left(2\alpha\right) - F_2\sin\left(2\alpha\right), \ F_2' = F_2\cos\left(2\alpha\right) + F_1\sin\left(2\alpha\right).$$

Choosing $\mathbf{l} = (0, 0, 1)$ and selecting α to be a solution of the equations

$$F_1' = F_1 \cos(2\alpha) \,, \ F_2' = F_1 \sin(2\alpha) \,, \ F_3' = F_3 + \dot{\alpha} \,,$$

we obtain

$$
\begin{aligned}
F_1 &= F_1' \cos(2\alpha) + F_2' \sin(2\alpha) \,, \ F_3 = F_3' - \dot{\alpha} \,, \\
F_2 &= F_2' \cos(2\alpha) - F_1' \sin(2\alpha) = 0 \,,
\end{aligned}
\tag{22}
$$

so that the reduced external field $\mathbf{F}$ takes the form

$$\mathbf{F} = (F_1, 0, F_3) \,. \tag{23}$$

In addition, one can see that if V is a solution of the SE with the external field (23) then:

1. $U = (2)^{-1/2} (1 + i\sigma_1) V$ is a solution of the SE with the external field

$$\mathbf{F} = (F_1, F_3, 0) \,; \tag{24}$$

2. $U = \sigma_1 V$ is a solution of the SE with the external field

$$\mathbf{F} = (F_1, 0, -F_3) \,; \tag{25}$$

3. $U = \sigma_3 V$ is a solution of the SE with the external field

$$\mathbf{F} = (-F_1, 0, F_3) \,; \tag{26}$$

4. $U = \sigma_2 V$ is a solution of the SE with the external field

$$\mathbf{F} = (-F_1, 0, -F_3) \,; \tag{27}$$

5. $U = (2)^{-1/2} (\sigma_1 + \sigma_3) V$ is a solution of the SE with the external field

$$\mathbf{F} = (F_3, 0, F_1) \,. \tag{28}$$

2.3   Dirac-like equation

Consider the SE with the external field $\mathbf{F} = (F_1, 0, F_3)$. Let us represent the complex spinor V in this SE via two real spinors U and W,

$$V = U + iW \,. \tag{29}$$

The SE for V implies the following equations for the real spinors U and W:

$$\dot{U} = (\mathbf{G}\,\sigma)\,U + (\mathbf{K}\,\sigma)\,W \,, \ \dot{W} = (\mathbf{G}\,\sigma)\,W - (\mathbf{K}\,\sigma)\,U \,. \tag{30}$$

The latter set can be written as

$$\dot{\Psi} = [(\Sigma\,\mathbf{G}) + (\gamma\,\mathbf{K})]\,\Psi \,, \tag{31}$$

where Σ and γ are Dirac matrices, and Ψ is a four-component spinor:

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}, \ \gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \ \Psi = \begin{pmatrix} U \\ W \end{pmatrix}. \tag{32}$$

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2.4 Top equation

If V is a solution of the SE, then the vectors (138) (see Appendix) obey the following equations:

$$\dot{\mathbf{L}}^{v,v} = i\left(\mathbf{F}^* - \mathbf{F}\right)(V,V) + \left[(\mathbf{F} + \mathbf{F}^*) \times \mathbf{L}^{v,v}\right],$$
$$\dot{\mathbf{L}}^{\bar{v},v} = 2\left[\mathbf{F} \times \mathbf{L}^{\bar{v},v}\right],\ \dot{\mathbf{L}}^{v,\bar{v}} = 2\left[\mathbf{F}^* \times \mathbf{L}^{v,\bar{v}}\right]. \tag{33}$$

At the same time, the following relations hold:

$$\mathbf{L}^{v,\dot{v}} = -i\left(V,V\right)\mathbf{F} + \left[\mathbf{F} \times \mathbf{L}^{v,v}\right],\ \mathbf{L}^{\bar{v},\dot{v}} = \mathbf{L}^{\dot{\bar{v}},v} = \left[\mathbf{F} \times \mathbf{L}^{\bar{v},v}\right]. \tag{34}$$

In addition, the vectors (140) (see Appendix) obey the following equations:

$$\dot{\mathbf{e}}_1 = 2\mathbf{e}_2\left(\mathbf{Kn}\right) - 2\mathbf{n}\left(\mathbf{Kn} + \mathbf{Ge}_1\right),$$
$$\dot{\mathbf{e}}_2 = 2\mathbf{n}\left(\mathbf{Ke}_1 - \mathbf{Ge}_2\right) - 2\mathbf{e}_1\left(\mathbf{Kn}\right),$$
$$\dot{\mathbf{n}} = 2\mathbf{e}_1\left(\mathbf{Ke}_2 + \mathbf{Ge}_1\right) + 2\mathbf{e}_2\left(\mathbf{Ge}_2 - \mathbf{Ke}_1\right), \tag{35}$$

with $\mathbf{K}$ and $\mathbf{G}$ given by (13). Supposing that V obeys the SE, we can find the equations of motion for the parameters $N, \alpha, \theta,$ and φ from the representation (143). Taking into account (144) and (131), we get

$$2\dot{V} = \left(2N^{-1}\dot{N} + i\dot{\alpha} - i\dot{\varphi}\cos\theta\right)V + \left(\dot{\theta} + i\dot{\varphi}\sin\theta\right)\exp\left(i\alpha\right)\bar{V}. \tag{36}$$

Then, with the help of (36), (147) and (13), we finally obtain

$$\dot{\theta} = 2\mathbf{Ke}_\varphi + 2\mathbf{Ge}_\theta,\ \dot{\varphi}\sin\theta = 2\mathbf{Ge}_\varphi - 2\mathbf{Ke}_\theta, \tag{37}$$
$$\dot{\alpha} = \dot{\varphi}\cos\theta - 2\mathbf{Kn},\ \dot{N} = N\mathbf{Gn}. \tag{38}$$

The set (37) is autonomous (since it does not contain the functions N, α) and can be written in the compact form

$$\dot{\mathbf{n}} = 2\left[\mathbf{G} - \left(\mathbf{Gn}\right)\mathbf{n}\right] + 2\left[\mathbf{K} \times \mathbf{n}\right]. \tag{39}$$

Thus, the time evolution of the vector $\mathbf{n}$ is determined by the external field only. If the set (39) can be integrated to obtain $\theta\left(t\right)$ and $\varphi\left(t\right)$, then we can find from (38)

$$\alpha = \int \left(\dot{\varphi}\cos\theta - 2\mathbf{Kn}\right)dt,\ N = \exp\left(\int \mathbf{Gn}\,dt\right). \tag{40}$$

Equation (39) for $\mathbf{G} = 0$ is the well-known top equation. It appears in the gyroscope theory, in the theory of precession of a classical gyromagnet in a magnetic field, in the theory of electromagnetic resonance (see [5]), and so on. For $\mathbf{G} \neq 0$, this equation describes a possible damping of the system.

3 General solution of the spin equation

The general solution Y_{gen} of the SE can always be written as

$$Y_{\text{gen}} = aV + bU\,, \tag{41}$$

where a and b are arbitrary complex constants, while $V(t)$ and $U(t)$ are two linearly-independent particular solutions of the SE. In fact, one needs to know only one particular solution V, since the other solution U can be constructed from V in quadratures. Indeed, according to (131), one can always present U in the form

$$U = \alpha V + \beta \bar{V}\,, \tag{42}$$

where $\alpha(t)$ and $\beta(t)$ are complex functions of time. Substituting (42) into the SE, and taking into account (12), (13), we find

$$\dot{\alpha}V + \dot{\beta}\bar{V} = 2\beta\sigma\mathbf{G}\bar{V}\,. \tag{43}$$

Hence, multiplying this relation by V^+ and $\bar{V}^+$, we obtain, with allowance for (135) and (136),

$$(V,V)\dot{\beta} = -2\beta\mathbf{L}^{v,v}\mathbf{G}\,, \ \ \dot{\alpha} = 2\beta(V,V)^{-1}\mathbf{L}^{v,\overline{v}}\mathbf{G}\,. \tag{44}$$

Taking into account (146) and (38), one can rewrite the first of these equations in the form

$$N\dot{\beta} = -2\dot{N}\beta\,. \tag{45}$$

Equation (45) can be easily integrated:

$$\beta = \beta_0 N^{-2} = \beta_0(V,V)^{-1}\,, \tag{46}$$

where β_0 is an arbitrary complex constant. Then, the second equation in (44) implies

$$\dot{\alpha} = 2\beta_0(V,V)^{-2}\mathbf{L}^{v,\overline{v}}\mathbf{G}\,. \tag{47}$$

Hence, α can be found by integration:

$$\alpha = \alpha_0 + 2\beta_0\int(V,V)^{-2}\mathbf{L}^{v,\overline{v}}\mathbf{G}dt\,, \tag{48}$$

where α_0 is an arbitrary complex constant. Thus, the general solution $Y_{\text{gen}}(t)$ of the SE, with the known particular solution V of this equation, has the form

$$Y_{\text{gen}} = \left[\alpha_0 + 2\beta_0\int(V,V)^{-2}\mathbf{L}^{v,\overline{v}}\mathbf{G}dt\right]V + \beta_0(V,V)^{-1}\bar{V}\,, \tag{49}$$

where α_0, β_0 are arbitrary complex constants.

4 Stationary solutions

Consider the SE with a constant external filed, $\mathbf{F} = $ const. In this case, we can search for stationary solutions of the form

$$V(t) = \exp(-i\lambda t) V, \tag{50}$$

where V is a time-independent spinor subject to the equation

$$(\sigma\mathbf{F})V = \lambda V. \tag{51}$$

This equations is analyzed in the Appendix, see below. In particular, for $\mathbf{F}^2 \neq 0$, we have two independent solutions V_ζ, λ_ζ, $\zeta = \pm 1$,

$$V_1 = N_1 \begin{pmatrix} F_3 + \sqrt{\mathbf{F}^2} \\ iF_2 + F_1 \end{pmatrix}, \; \lambda_1 = \sqrt{\mathbf{F}^2},$$

$$V_{-1} = N_{-1} \begin{pmatrix} iF_2 - F_1 \\ F_3 + \sqrt{\mathbf{F}^2} \end{pmatrix}, \; \lambda_2 = -\sqrt{\mathbf{F}^2}, \tag{52}$$

where N_ζ are normalization factors.

5 The inverse problem for the spin equation

The inverse problem for the SE can be formulated as follows: provided that a solution V of the SE is known, is it possible to recover the external field $\mathbf{F}$ using this solution? The answer to this question can be found in the general case.

A spinor V gives rise to a triplet of linear vectors (138). Let us decompose the external field in these vectors:

$$\mathbf{F} = c_1 \mathbf{L}^{v,v} + c_2 \mathbf{L}^{v,\bar{v}} + c\mathbf{L}^{\bar{v},v}, \tag{53}$$

where c_1, c_2, and c are some time-dependent coefficients. Substituting this expression into the SE and using formula (137), with allowance for (131), we find

$$i\dot{V} = (\sigma\mathbf{F})V = (V,V)\left(c_1 V + 2c_2 \bar{V}\right). \tag{54}$$

Multiplying this relation from the left by V^+ and $\bar{V}^+$, we obtain

$$c_1 = \frac{i\left(V, \dot{V}\right)}{(V,V)^2}, \; c_2 = \frac{i\left(\bar{V}, \dot{V}\right)}{2(V,V)^2}. \tag{55}$$

Substituting (55) into (53), we finally have

$$\mathbf{F} = \frac{i}{2(V,V)^2} \left[2\left(V, \dot{V}\right)\mathbf{L}^{v,v} + \left(\bar{V}, \dot{V}\right)\mathbf{L}^{v,\bar{v}}\right] + c\mathbf{L}^{\bar{v},v}. \tag{56}$$

Here, the complex function $c(t)$ remains arbitrary. Thus, there exist an infinite number of external fields $\mathbf{F}$ which admit the same solution of the SE, and the corresponding functional arbitrariness is completely described.

One can write (56) in a different form:

$$\mathbf{F} = \frac{i}{2\,(V,V)^2} \left\{ (V,V)\left(\mathbf{L}^{v,\dot{v}} - \mathbf{L}^{\dot{v},v}\right) + \left[\left(V,\dot{V}\right) + \left(\dot{V},V\right)\right]\mathbf{L}^{v,v} \right\} + b\mathbf{L}^{\overline{v},v}\,. \quad (57)$$

Here, $b\,(t)$ is also an arbitrary complex function. Expression (57) can be easily reduced to (56) with allowance for (139).

The functional arbitrariness arising in the solution of the inverse problem is related to the fact that the spinor V is given by two complex functions, whereas the external field $\mathbf{F}$ is defined by three complex functions.

In this way, we have also demonstrated that any complex spinor with an arbitrary time-dependence (provided that this spinor is differentiable) is a solution of a certain family of the SE.

Taking into account the explicit form (143) of the spinor V and using formulas (146) and (36), one easily deduces from (56) that

$$\mathbf{F} = \frac{1}{2}\left[(\dot{\varphi}\cos\theta - \dot{\alpha})\,\mathbf{n} - \dot{\varphi}\mathbf{e}_\theta\sin\theta + \dot{\theta}\mathbf{e}_\varphi\right] + i\frac{\dot{N}}{N}\mathbf{n} + a\,(\mathbf{e}_\theta + i\mathbf{e}_\varphi)\,, \qquad (58)$$

where $a\,(t)$ is an arbitrary complex function.

6 Self-adjoint spin equation

6.1 General solution and inverse problem

We shall refer to the SE as self-adjoint if the external field $\mathbf{F}$ is real. In this case, according to (13), we have

$$\mathbf{F} = \operatorname{Re}\mathbf{F} = \mathbf{K}\,,\ \ \operatorname{Im}\mathbf{F} = \mathbf{G} = 0\,. \qquad (59)$$

In this case, the SE has the form of a Schrödinger equation,

$$i\dot{V} = HV\,,\ \ H = \sigma\mathbf{F} = H^+\,, \qquad (60)$$

with a Hermitian Hamiltonian H. Nevertheless, even for a self-adjoint SE, the one-dimensional Hamiltonians (16) are not Hermitian in the general case.

Below, we list some properties of a self-adjoint SE, which, generally speaking, do not take place for a generic complex external field.

The general solution Y_{gen} of a self-adjoint SE has the form

$$Y_{\mathrm{gen}} = aV + b\bar{V}\,, \qquad (61)$$

where $V\,(t)$ is any nonzero particular solution of the SE, and a, b are arbitrary complex constants. This fact follows from (44).

For any solution V of a self-adjoint SE, the quantity $N^2 = (V,V)$ is conserved in time, which is implied by (38) in case $\mathbf{G} = 0$. However, the reverse is not true. The fact that $N^2 = \mathrm{const}$ does not imply that V is a solution of a self-adjoint

SE, since one can indicate, according to (56), a family of complex external fields of the SE that admit solutions with $N^2 = \text{const}$.

For an arbitrary nonzero differentiable spinor V subject to the condition $(V, V) = \text{const}$, there exists only one self-adjoint SE (only one real external field), whose solution is given by this spinor, and whose general solution has the form (61). Indeed, it follows from (57) that in this case a real external field is recovered by the spinor V in a unique manner:

$$\mathbf{F} = i \left[2 \left(V, V \right) \right]^{-1} \left(\mathbf{L}^{v,\dot{v}} - \mathbf{L}^{\dot{v},v} \right) . \tag{62}$$

It can be easily verified that the same expression for $\mathbf{F}$ arises in the case when V is replaced by Y_{gen} from (61), which confirms the uniqueness of the external field $\mathbf{F}$. Now, presenting V in the form (143) (see Appendix), and setting $N = \text{const}$, one obtains a decomposition of $\mathbf{F}$ in the basis vectors of a spherical coordinate system (145):

$$\mathbf{F} = \frac{1}{2} \left[\left(\dot{\varphi} \cos \theta - \dot{\alpha} \right) \mathbf{n} - \dot{\varphi} \mathbf{e}_\theta \sin \theta + \dot{\theta} \mathbf{e}_\varphi \right] . \tag{63}$$

Hence, one can find the Cartesian components of the external field and calculate its square:

$$\mathbf{F} = \frac{1}{2} (-\dot{\theta} \sin \varphi - \dot{\alpha} \sin \theta \cos \varphi, \ \dot{\theta} \cos \varphi - \dot{\alpha} \sin \theta \sin \varphi, \ \dot{\varphi} - \dot{\alpha} \cos \theta) ,$$

$$F^2 = \mathbf{F}^2 = \frac{1}{4} \left(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\alpha}^2 - 2 \dot{\alpha} \dot{\varphi} \cos \theta \right) . \tag{64}$$

The possibility of an unambiguous recovery of the real external field $\mathbf{F}$ by a given arbitrary spinor $V(t)$ with a constant norm also signifies the possibility of generating exactly solvable self-adjoint SE.

For equations that are associated with a self-adjoint SE, one can state some additional properties. For instance, the evolution equations (33) for the linearly-independent vectors (138) become coincident, so that the vectors (138) have to be distinguished by an appropriate choice of initial conditions.

6.2 Hamiltonian and Lagrangian forms of self-adjoint spin equation

Consider the set of equations (37) for real external fields. In this case, the set can be written as

$$\dot{\theta} = 2 \left(F_2 \cos \varphi - F_1 \sin \varphi \right) , \ \ \dot{\varphi} \sin \theta = 2 F_3 \sin \theta - 2 \left(F_1 \cos \varphi + F_2 \sin \varphi \right) \cos \theta . \tag{65}$$

Without loss of generality, one can always choose

$$F_1 = g(t) \cos \left[2 \alpha(t) \right] , \ \ F_2 = g(t) \sin \left[2 \alpha(t) \right] , \tag{66}$$

where $g(t)$ and $\alpha(t)$ are some real functions of time. Let us replace $\varphi(t)$ in equation (65) by a new function $\Phi(t)$, introduced as

$$\varphi(t) = \Phi(t) + 2 \alpha(t) . \tag{67}$$

It is easy to see that the set (65) transforms to

$$\dot{\theta} = -2g\sin\Phi\,,\;\; \dot{\Phi}\sin\theta = 2f\sin\theta - 2g\cos\Phi\cos\theta\,, \tag{68}$$

where

$$f = F_3(t) - \dot{\alpha}(t)\,. \tag{69}$$

Notice that the replacement (67) is equivalent to the transformation (17), if one chooses $\mathbf{e} = (0,0,1)$ and selects $\alpha(t)$ such that the external field takes the form (23).

If one introduces the coordinate q, the conjugate momentum p and the Hamiltonian H as follows

$$q = \cos\theta\,,\;\; p = -\Phi\,,\;\; H = 2g\sqrt{1-q^2}\cos p + 2qf\,, \tag{70}$$

then the set (68) takes the form of one-dimensional Hamilton equations [5, 12],

$$\dot{q} = \frac{\partial H}{\partial p}\,,\;\; \dot{p} = -\frac{\partial H}{\partial q}\,. \tag{71}$$

Making canonical transformations, we can obtain different forms of the Hamilton equations that are associated with the self-adjoint SE.

It is straightforward to check that (68) are the Euler–Lagrange equations for the Lagrange function

$$\mathcal{L} = \left[(1-\gamma)\dot{\theta}\Phi - 2g\cos\Phi\right]\sin\theta + \left[\gamma\dot{\Phi} - 2f\right]\cos\theta\,, \tag{72}$$

where γ is an arbitrary real number.

Finally, the set (68) leads to a second-order equation for the function $\theta(t)$:

$$\ddot{\theta} - \frac{\dot{g}}{g}\dot{\theta} + 2f\sqrt{4g^2 - \dot{\theta}^2} - \left(4g^2 - \dot{\theta}^2\right)\frac{\cos\theta}{\sin\theta} = 0\,. \tag{73}$$

This equation is also the Euler–Lagrange equation for the Lagrange function

$$\mathcal{L} = \dot{\theta}\arcsin\left(\dot{\theta}/2g\right)\sin\theta + \sqrt{4g^2 - \dot{\theta}^2}\sin\theta + 2f\cos\theta\,. \tag{74}$$

The above Lagrangian implies the following Hamiltonian

$$H = -2\left[g\cos\left(\frac{p}{\sin\theta}\right)\sin\theta + f\cos\theta\right]\,,\;\; p = \arcsin\left(\dot{\theta}/2g\right)\sin\theta \tag{75}$$

which, from the canonical equation (71), gives the equation (73).

7   More about solutions of the spin equation

7.1   The transformation matrix

Suppose that we know a solution V_1 of the SE with an external field $\mathbf{F}_1$,

$$i\dot{V}_1 = (\sigma\mathbf{F}_1)\,V_1\,,$$

and wish to find such a nonsingular time-dependent matrix $\hat{T}^{21}$ (in what follows, it is called the transformation matrix) that a spinor V_2,

$$V_2(t) = \hat{T}^{21}(t) V_1(t) \,, \tag{76}$$

is a solution of the SE with an external field $\mathbf{F}_2$,

$$i\dot{V}_2 = (\sigma \mathbf{F}_2) V_2 \,. \tag{77}$$

It is easy to obtain an equation for the transformation matrix:

$$i\frac{d}{dt}\hat{T}^{21} = (\sigma \mathbf{F}_2)\hat{T}^{21} - \hat{T}^{21}(\sigma \mathbf{F}_1) \,. \tag{78}$$

Like any 2×2 matrix, the matrix $\hat{T}^{21}$ can be written in the form

$$\hat{T}^{21} = a_0 - i\sigma\mathbf{a}\,, \ \mathbf{a} = (a_1, a_2, a_3) \,, \tag{79}$$

where $a_s(t)$, $s = 0, 1, 2, 3$ are some complex functions of t. Substituting (79) into (78), and using elementary properties of the Pauli matrices, one obtains the following set of equations for the functions a_s:

$$\dot{a}_0 + \mathbf{a}\mathbf{F}_{21} = 0\,, \ \mathbf{F}_{21} = \mathbf{F}_2 - \mathbf{F}_1\,,$$
$$\dot{\mathbf{a}} + 2[\mathbf{a} \times \mathbf{F}_1] + [\mathbf{a} \times \mathbf{F}_{21}] - a_0\mathbf{F}_{21} = 0\,. \tag{80}$$

It is easy to find that $\Delta = \det \hat{T}^{21} = a_0^2 + \mathbf{a}^2$ is an integral of motion. Since the matrix $\hat{T}^{21}$ is determined by (78) only with accuracy up to a constant complex multiplier, we choose, without loss of generality,

$$\Delta = a_0^2 + \mathbf{a}^2 = 1\,. \tag{81}$$

For the inverse matrix $\left(\hat{T}^{21}\right)^{-1}$, we obtain

$$\left(\hat{T}^{21}\right)^{-1} = \Delta^{-1}(a_0 + i\sigma\mathbf{a}) = a_0 + i\sigma\mathbf{a}\,. \tag{82}$$

Given $\mathbf{F}_1$ and $\mathbf{F}_2$, equations (80) are a linear homogenous (complex) set of four ordinary differential equations of first order. Solving this set is completely analogous to solving the SE with the external field $\mathbf{F}_2$, so that we do not achieve any simplification. However, assuming that the external field $\mathbf{F}_1$ and the matrix $\hat{T}^{21}$ are known, we can obtain from (80) the external field $\mathbf{F}_2$. It turns out that this problem can be easily solved. To this end, we need to consider two cases:

1. Let $a_0 \neq 0$. We introduce a complex vector $\mathbf{q} = \mathbf{q}(t)$,

$$\mathbf{q} = \mathbf{a}/a_0 \,, \tag{83}$$

such that $\mathbf{q}^2 \neq -1$. Then (81) implies

$$a_0 = \left(1 + \mathbf{q}^2\right)^{-1/2} \,. \tag{84}$$

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From (80), one obtains the equation

$$\dot{\mathbf{q}} - \mathbf{q}\,(\mathbf{q}\mathbf{F}_{21}) + [\mathbf{q} \times \mathbf{F}_{21}] + 2\,[\mathbf{q} \times \mathbf{F}_1] - \mathbf{F}_{21} = 0\,. \tag{85}$$

This equation allows one to find a unique representation for the external field $\mathbf{F}_2$,

$$\mathbf{F}_2 = \frac{\dot{\mathbf{q}} + [\mathbf{q} \times \dot{\mathbf{q}}] + 2\,[\mathbf{q} \times \mathbf{F}_1] + 2\mathbf{q}\,(\mathbf{q}\mathbf{F}_1) - 2\mathbf{q}^2\mathbf{F}_1}{1 + \mathbf{q}^2} + \mathbf{F}_1\,. \tag{86}$$

In this case, the transformation matrix reads

$$\hat{T}^{21} = \frac{1 - i\sigma\mathbf{q}}{\sqrt{1 + \mathbf{q}^2}}\,, \quad \left(\hat{T}^{21}\right)^{-1} = \frac{1 + i\sigma\mathbf{q}}{\sqrt{1 + \mathbf{q}^2}}\,. \tag{87}$$

2. Let $a_0 = 0$. We introduce a vector $\mathbf{q} = \mathbf{q}\,(t)$,

$$\mathbf{q} = \mathbf{a}\,.$$

In this case, the condition (81) implies $\mathbf{q}^2 = 1$, and we obtain from (80)

$$\dot{\mathbf{q}} + [\mathbf{q} \times \mathbf{F}_{21}] + 2\,[\mathbf{q} \times \mathbf{F}_1] = 0\,, \quad \mathbf{q}\mathbf{F}_{21} = 0\,. \tag{88}$$

From (88), we uniquely recover $\mathbf{F}_2$ in the form

$$\mathbf{F}_2 = [\mathbf{q} \times \dot{\mathbf{q}}] + 2\mathbf{q}\,(\mathbf{q}\mathbf{F}_1) - \mathbf{F}_1\,, \quad \mathbf{q}^2 = 1\,. \tag{89}$$

The transformation matrix now reads

$$\hat{T}^{21} = -\left(\hat{T}^{21}\right)^{-1} = -i\sigma\mathbf{q}\,. \tag{90}$$

Consequently, having an exact solution V_0 that corresponds to the external field $\mathbf{F}_1$, we can construct a family of external fields $\mathbf{F}_2$ and the corresponding exact solutions (76), parametrized by an arbitrary complex time-dependent vector $\mathbf{q}$.

In the particular case of a self-adjoint SE, the above statement remains valid if one assumes $\mathbf{q}$ to be a real vector. In this case, the transformation matrix is unitary.

7.2 Evolution operator for the spin equation

In the above consideration, let us choose $\mathbf{F}_1 = 0$ and denote $\mathbf{F}_2 = \mathbf{F}$, $\hat{T}^{21} = \hat{T}$. Then one can select V_1 as an arbitrary constant spinor: $V_1 = V_0 = \text{const}$. For $\mathbf{F}_1 = 0$, one deduces from (78) that the transformation matrix $\hat{T}$ obeys the equation

$$i\frac{d\hat{T}}{dt} = (\sigma\mathbf{F})\,\hat{T}\,. \tag{91}$$

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If the transformation matrix $\hat{T}$ is known, the evolution operator $\hat{R}$, being a solution of equation (91) with the initial condition $\hat{R}(0) = I$, can be constructed as follows:

$$\hat{R}(t) = \hat{T}(t)\hat{T}^{-1}(0) \ . \tag{92}$$

Using the above expressions for the transformation matrix, we can construct the evolution operator for the SE with any external field $\mathbf{F}$ according to (92). The answer reads:

1. Let us select an arbitrary complex time-dependent vector $\mathbf{q}(t)$ $(\mathbf{q}(0) = \mathbf{q}_0)$ such that $\mathbf{q}^2 \neq -1$. Then the SE with the external field

$$\mathbf{F} = \frac{\dot{\mathbf{q}} + [\mathbf{q} \times \dot{\mathbf{q}}]}{1 + \mathbf{q}^2} \tag{93}$$

   has the evolution operator of the form

$$\hat{R} = \frac{(1 - i\sigma\mathbf{q})(1 + i\sigma\mathbf{q}_0)}{\sqrt{(1 + \mathbf{q}^2)(1 + \mathbf{q}_0^2)}} = \frac{1 + \mathbf{q}\mathbf{q}_0 - i\sigma\mathbf{p}}{\sqrt{(1 + \mathbf{q}^2)(1 + \mathbf{q}_0^2)}} \ , \tag{94}$$

   where $\mathbf{p} = \mathbf{q} - \mathbf{q}_0 + [\mathbf{q}_0 \times \mathbf{q}]$.

2. Let us select an arbitrary complex unit time-dependent vector $\mathbf{q}(t)$ $(\mathbf{q}(0) = \mathbf{q}_0)$, $\mathbf{q}^2 = 1$. Then the SE with the external field

$$\mathbf{F} = [\mathbf{q} \times \dot{\mathbf{q}}] \ , \tag{95}$$

   has the evolution operator of the form

$$\hat{R} = (\sigma\mathbf{q})(\sigma\mathbf{q}_0) = \mathbf{q}\mathbf{q}_0 + i\sigma[\mathbf{q} \times \mathbf{q}_0] \ . \tag{96}$$

In the case of a self-adjoint SE (real external fields), $\mathbf{q}$ is selected as a real vector, and the operator $\hat{R}$ is unitary.

8 Exact solutions of the spin equation

The first remark: Let $V(t)$ be a solution of the SE with a given external field $\mathbf{F}$. In this equation, we make the following change of the variable:

$$t = T(t') \ , \tag{97}$$

where t' is the new real variable (T is a real invertible function). Then the SE takes the form

$$i\frac{dV'(t)}{dt} = (\sigma\mathbf{F}'(t))V'(t) \ , \tag{98}$$

where

$$\mathbf{F}'(t) = \mathbf{F}(T(t))\dot{T} \ , \ V'(t) = V(T(t)) \ . \tag{99}$$

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Consequently, if one knows a solution of the SE with an external field $\mathbf{F}$, then one knows a solution of the SE with external fields $\mathbf{F}'$, parametrized by an arbitrary function T. In this sense, all solutions are divided into equivalence classes. Below, we are going to list only those solutions of the SE that belong to different classes.

The second remark: We have demonstrated that the SE with an arbitrary external field can be reduced to an equivalent SE with the external field (23) which has only two nonzero components:

$$\mathbf{F} = (F_1, 0, F_3) \,. \tag{100}$$

Below, we are going to list only solutions for such external fields.

The third remark: Let the components F_1 and F_3 of the external field be linearly dependent. Then, without loss of the generality, we can write

$$F_1 = q \sin \lambda \,, \ F_3 = q \cos \lambda \,, \tag{101}$$

where $q(t)$ is an arbitrary function of time, and λ is a complex constant. Let us define the function $\omega(t)$ by the relations

$$\dot{\omega} = q \,, \ \omega(0) = 0 \,. \tag{102}$$

Then, the evolution operator for the SE with such an external field has the form

$$\hat{R} = \cos \omega - i \left(\sigma_1 \sin \lambda + \sigma_3 \cos \lambda \right) \sin \omega t \,. \tag{103}$$

Especially interesting are solutions of the SE that can be written via the known special functions.

Below, we consider external fields with such nonzero components F_1 and F_3 that obey the following properties: if solutions of the SE are known for such external fields, then one can construct solutions for the external fields

$$\mathbf{F} = (\alpha F_1(\varphi)\,, 0, \beta F_3(\varphi))\,, \ \varphi = \omega t + \varphi_0 \,, \tag{104}$$

where α, β, and φ_0 are arbitrary real constants. We have succeeded in finding 26 pairs of linearly independent functions F_1 and F_3 that conform to this condition. Below, we list such pairs and present the spinor u being the corresponding exact solution of the SE. We use the following notation [13]:

$F(\alpha, \beta; \gamma; z)$ is the Gauss hypergeometric function;

$\Phi(\alpha, \gamma; z)$ is the degenerate hypergeometric function;

$D_p(z)$ are the parabolic cylinder functions;

$\varphi = \omega t + \varphi_0$;

ω and φ_0 are real constants;

a, b, c and $\alpha, \beta, \gamma, \lambda, \mu, \nu$ are complex constants.

8.1 List of exact solutions

1. $F_1 = at$, $F_3 = bt + c/t$:

$$u = \begin{pmatrix} at^{\gamma+2}e^{-z/2}\Phi\left(\alpha+1,\gamma+2;z\right) \\ 2\left(i-c\right)t^{\gamma}e^{-z/2}\Phi\left(\alpha,\gamma;z\right) \end{pmatrix},$$

$$z = it^2\sqrt{a^2+b^2}\,,\ \alpha = \frac{\gamma}{2}\left(1+\frac{b}{\sqrt{a^2+b^2}}\right),\ \gamma = ic\,.$$

2. $F_1 = a/t$, $F_3 = b/t + ct$:

$$u = \begin{pmatrix} -at^{\gamma-1}e^{-z/2}\Phi\left(\alpha,\gamma;z\right) \\ \left(\sqrt{a^2+b^2}+b\right)t^{\gamma-1}e^{-z/2}\Phi\left(\alpha+1,\gamma;z\right) \end{pmatrix},$$

$$z = ict^2\,,\ 2\alpha = i\left(\sqrt{a^2+b^2}+b\right),\ \gamma = 1+i\sqrt{a^2+b^2}\,.$$

3. $F_1 = a/t$, $F_3 = b/t + c$:

$$u = \begin{pmatrix} -at^{(\gamma-1)/2}e^{-z/2}\Phi\left(\alpha,\gamma;z\right) \\ -iat^{(\gamma-1)/2}e^{-z/2}\Phi\left(1+\alpha,\gamma;z\right) \end{pmatrix},$$

$$z = 2ict\,,\ \alpha = i\left(\sqrt{a^2+b^2}+b\right),\ \gamma = 1+2i\sqrt{a^2+b^2}\,.$$

4. $F_1 = a/\sin 2\varphi$, $F_3 = \left(b\cos 2\varphi + c\right)/\sin 2\varphi$:

$$u = \begin{pmatrix} -az^{\mu}\left(1-z\right)^{\nu}F\left(\alpha+1,\beta;\gamma;z\right) \\ \left(-4i\omega\mu+b+c\right)z^{\mu}\left(1-z\right)^{\nu}F\left(\alpha,\beta+1;\gamma;z\right) \end{pmatrix},$$

$$z = \sin^2\varphi\,,\ \mu = \frac{i}{4\omega}\sqrt{a^2+\left(b+c\right)^2}\,,\ \nu = \frac{i}{4\omega}\sqrt{a^2+\left(b-c\right)^2}\,,$$

$$\alpha = \mu+\nu-ib/2\omega\,,\ \beta = \mu+\nu+ib/2\omega\,,\ \gamma = 1+2\mu\,.$$

5. $F_1 = a\tan\varphi$, $F_3 = b\tan\varphi + c\cot\varphi$:

$$u = \begin{pmatrix} 2\left(c+i\omega\right)z^{\mu}\left(1-z\right)^{\nu}F\left(\alpha,\beta;2\mu;z\right) \\ az^{\mu+1}\left(1-z\right)^{\nu}F\left(\alpha+1,\beta+1;2\mu+2;z\right) \end{pmatrix},$$

$$z = \sin^2\varphi\,,\ \mu = -\frac{ic}{2\omega}\,,\ \nu = \frac{i}{2\omega}\sqrt{a^2+b^2}\,,$$

$$\lambda = \frac{i}{2\omega}\sqrt{a^2+\left(b-c\right)^2}\,,\ \alpha = \nu+\mu+\lambda\,,\ \beta = \nu+\mu-\lambda\,.$$

6. $F_1 = a/\sin\varphi$, $F_3 = b\tan\varphi + c\cot\varphi$:

$$u = \begin{pmatrix} -az^{\mu}\left(1-z\right)^{\nu+1/2}F\left(\alpha+1,\beta;2\mu+1;z\right) \\ \left(\sqrt{a^2+c^2}+c\right)z^{\mu}\left(1-z\right)^{\nu}F\left(\alpha,\beta;2\mu+1;z\right) \end{pmatrix},$$

$$\mu = \frac{i}{2\omega}\sqrt{a^2+c^2}\,,\ \nu = -\frac{ib}{2\omega}\,,\ z = \sin^2\varphi\,,$$

$$\alpha = \mu-\frac{ic}{2\omega}\,,\ \beta = \frac{1}{2}+\mu+2\nu+\frac{ic}{2\omega}\,.$$

7. $F_1 = a/\cos\varphi$, $F_3 = b\tan\varphi + c$:

$$u = \begin{pmatrix} (\omega + 2c - 2ib)\, z^\mu (1-z)^\nu\, F(\alpha, \beta; \gamma; z) \\ 2ia z^{\mu+1/2}(1-z)^\nu\, F(\alpha, \beta+1; \gamma+1; z) \end{pmatrix},$$

$$z = -e^{-2i\varphi}, \ \mu = \frac{c - ib}{2\omega}, \ \nu = \frac{i}{\omega}\sqrt{a^2 + b^2},$$

$$\alpha = \frac{1}{2} + \frac{c}{\omega} + \nu, \ \beta = \nu - \frac{ib}{\omega}, \ \gamma = \frac{1}{2} + 2\mu.$$

8. $F_1 = a/\sinh\varphi$, $F_3 = b\tanh\varphi + c\coth\varphi$:

$$u = \begin{pmatrix} -a z^\mu (1-z)^\nu\, F(\alpha, \beta; \gamma; z) \\ (-2i\omega\mu a + c)\, z^\mu (1-z)^{\nu+1/2}\, F(\alpha, \beta+1; \gamma; z) \end{pmatrix},$$

$$z = \tanh^2\varphi, \ \mu = \frac{i}{2\omega}\sqrt{a^2 + c^2}, \ \nu = \frac{i(b+c)}{2\omega},$$

$$\alpha = \frac{1}{2} + \frac{ib}{\omega} + \beta, \ \beta = \mu + \frac{ic}{2\omega}, \ \gamma = 2\mu + 1.$$

9. $F_1 = a/\cosh\varphi$, $F_3 = b\tanh\varphi + c\coth\varphi$:

$$u = \begin{pmatrix} (2c + i\omega) z^\mu (1-z)^\nu\, F(\alpha, \beta; \gamma; z) \\ a z^{\mu+1/2}(1-z)^{\nu+1/2}\, F(\alpha+1, \beta+1; \gamma+1; z) \end{pmatrix},$$

$$z = \tanh^2\varphi, \ \mu = -\frac{ic}{2\omega}, \ \nu = \frac{i(b+c)}{2\omega}, \ \lambda = \frac{1}{2\omega}\sqrt{a^2 - b^2},$$

$$\alpha = \frac{ib}{2\omega} + \lambda, \ \beta = \frac{ib}{2\omega} - \lambda, \ \gamma = \frac{1}{2} - \frac{ic}{\omega}.$$

10. $F_1 = a/\sinh 2\varphi$, $F_3 = (b\cosh 2\varphi + c)/\sinh 2\varphi$:

$$u = \begin{pmatrix} -a z^\mu (1-z)^\nu\, F(\alpha, \beta; \gamma; z) \\ (-4i\omega\mu + b + c)\, z^\mu (1-z)^{\nu+1}\, F(\alpha+1, \beta+1; \gamma; z) \end{pmatrix},$$

$$z = \tanh^2\varphi, \ \mu = \frac{i}{4\omega}\sqrt{a^2 + (b+c)^2}, \ \lambda = \frac{i}{4\omega}\sqrt{a^2 + (b-c)^2},$$

$$\nu = \frac{ib}{2\omega}, \ \alpha = \mu + \nu + \lambda, \ \beta = \mu + \nu - \lambda, \ \gamma = 1 + 2\mu.$$

11. $F_1 = a/\cosh\varphi$, $F_3 = (b\sinh\varphi + c)/\cosh\varphi$:

$$u = \begin{pmatrix} a z^\mu (1-z)^\nu\, F(\alpha, \beta; \gamma; z) \\ (2\omega\mu - c + ib)\, z^\mu (1-z)^{\nu+1}\, F(\alpha+1, \beta+1; \gamma; z) \end{pmatrix},$$

$$z = \left(\frac{e^\varphi + i}{e^\varphi - i}\right)^2, \ \mu = \frac{1}{2\omega}\sqrt{a^2 + (c - ib)^2}, \ \alpha = \mu + \nu + \lambda,$$

$$\lambda = \frac{1}{2\omega}\sqrt{a^2 + (c + ib)^2}, \ \nu = \frac{ib}{\omega}, \ \beta = \mu + \nu - \lambda, \ \gamma = 1 + 2\mu.$$

12. $F_1 = a \tanh \varphi$, $F_3 = b \tanh \varphi + c \coth \varphi$:

$$u = \left(\begin{array}{c} 2(c + i\omega)z^\mu (1 - z)^\nu F(\alpha, \beta; \gamma; z) \\ az^{\mu+1} (1 - z)^\nu F(\alpha + 1, \beta + 1; \gamma + 2; z) \end{array} \right),$$

$$z = \tanh^2 \varphi, \ \mu = -\frac{ic}{2\omega}, \ \nu = \frac{i}{2\omega}\sqrt{a^2 + (b + c)^2},$$

$$\lambda = \frac{i}{2\omega}\sqrt{a^2 + b^2}, \ \alpha = \mu + \nu + \lambda, \ \beta = \mu + \nu - \lambda, \ \gamma = 2\mu.$$

13. $F_1 = a \coth \varphi$, $F_3 = b \tanh \varphi + c \coth \varphi$:

$$u = \left(\begin{array}{c} -az^\mu (1 - z)^\nu F(\alpha + 1, \beta; \gamma; z) \\ (2\omega\mu + c) z^\mu (1 - z)^\nu F(\alpha, \beta + 1; \gamma; z) \end{array} \right),$$

$$z = \tanh^2 \varphi, \ \mu = \frac{i}{2\omega}\sqrt{a^2 + c^2}, \ \nu = \frac{i}{2\omega}\sqrt{a^2 + (b + c)^2},$$

$$\alpha = \mu + \nu + \frac{ib}{2\omega}, \ \beta = \mu + \nu - \frac{ib}{2\omega}, \ \gamma = 1 + 2\mu.$$

14. $F_1 = a/\cosh \varphi$, $F_3 = b \tanh \varphi + c$:

$$u = \left(\begin{array}{c} (2b + 2c - i\omega)z^\mu (1 - z)^\nu F(\alpha, \beta; \gamma; z) \\ 2az^{\mu+1/2} (1 - z)^{\nu+1/2} F(\alpha + 1, \beta + 1, \gamma + 1; z) \end{array} \right),$$

$$z = \frac{1}{2}(1 - \tanh \varphi), \ \alpha = \mu + \nu + \lambda, \ \beta = \mu + \nu - \lambda,$$

$$\mu = \frac{i(b + c)}{2\omega}, \ \nu = \frac{i(b - c)}{2\omega}, \ \gamma = 1/2 + 2\mu, \ \lambda = \frac{1}{\omega}\sqrt{a^2 - b^2}.$$

15. $F_1 = a/\sinh \varphi$, $F_3 = b \coth \varphi + c$:

$$u = \left(\begin{array}{c} -az^\mu (1 - z)^\nu F(\alpha, \beta; \gamma; z) \\ (-i\omega\mu + b) z^\mu (1 - z)^{\nu+1/2} F(\alpha, \beta + 1; \gamma; z) \end{array} \right),$$

$$z = 1 - e^{-2\varphi}, \ \mu = \frac{i}{\omega}\sqrt{a^2 + b^2}, \ \nu = \frac{i(b + c)}{2\omega},$$

$$\alpha = \frac{1}{2} + \mu + \frac{ic}{\omega}, \ \beta = \mu + \frac{ib}{\omega}, \ \gamma = 1 + 2\mu.$$

16. $F_1 = a$, $F_3 = bt + c$:

$$u = \left(\begin{array}{c} 2\sqrt{b}D_\mu(z) \\ (1 + i) aD_{\mu-1}(z) \end{array} \right),$$

$$z = \frac{1 + i}{\sqrt{b}}(bt + c), \ \mu = -\frac{ia^2}{2b}.$$

17. $F_1 = a$, $F_3 = b/t + c$:

$$u = \begin{pmatrix} (1 - 2ib)\, t^\gamma e^{-z/2} \Phi\left(\alpha, 2\gamma; z\right) \\ -iat^{\gamma+1} e^{-z/2} \Phi\left(\alpha + 1, 2\gamma + 2, z\right) \end{pmatrix},$$

$$z = 2it\sqrt{a^2 + c^2}\,,\ \gamma = -ib\,,\ \alpha = \gamma\left(1 - \frac{c}{\sqrt{a^2 + c^2}}\right).$$

18. $F_1 = a$, $F_3 = b/t + ct$:

$$u = \begin{pmatrix} (2b + i)t^{\gamma - 1/2} e^{-z/2} \Phi\left(\alpha, \gamma; z\right) \\ at^{\gamma+1/2} e^{-z/2} \Phi\left(\alpha + 1, \gamma + 1; z\right) \end{pmatrix},$$

$$z = ict^2\,,\ \alpha = \frac{ia^2}{4c}\,,\ \gamma = \frac{1}{2} - ib\,.$$

19. $F_1 = a$, $F_3 = \left(b\cos 2\varphi + c\right)/\sin 2\varphi$:

$$u = \begin{pmatrix} (b + c + i\omega)z^\mu \left(1 - z\right)^\nu F\left(\alpha, \beta; \gamma; z\right) \\ az^{\mu+1/2} \left(1 - z\right)^{\nu+1/2} F\left(\alpha + 1, \beta + 1; \gamma + 1; z\right) \end{pmatrix},$$

$$z = \sin^2 \varphi\,,\ \mu = -\frac{i}{4\omega}\left(b + c\right)\,,\ \nu = \frac{i}{4\omega}\left(c - b\right)\,,\ \gamma = \frac{1}{2} + 2\mu\,,$$

$$\alpha = \frac{1}{2\omega}\left(\sqrt{a^2 - b^2} - ib\right)\,,\ \beta = -\frac{1}{2\omega}\left(\sqrt{a^2 - b^2} + ib\right).$$

20. $F_1 = a$, $F_3 = b\tan\varphi + c\cot\varphi$:

$$u = \begin{pmatrix} (2c + i\omega)z^\mu \left(1 - z\right)^\nu F\left(\alpha, \beta; \gamma; z\right) \\ az^{\mu+1/2} \left(1 - z\right)^{\nu+1/2} F\left(\alpha + 1, \beta + 1; \gamma + 1; z\right) \end{pmatrix},$$

$$z = \sin^2 \varphi\,,\ \mu = -\frac{ic}{2\omega}\,,\ \nu = \frac{ib}{2\omega}\,,\ \lambda = \frac{1}{2\omega}\sqrt{a^2 - (b - c)^2}\,,$$

$$\alpha = \mu + \nu + \lambda\,,\ \beta = \mu + \nu - \lambda\,,\ \gamma = \frac{1}{2} + 2\mu\,.$$

21. $F_1 = a$, $F_3 = b\tan\varphi + c$:

$$u = \begin{pmatrix} az^\mu \left(1 - z\right)^\nu F\left(\alpha, \beta; \gamma; z\right) \\ (2\omega\mu - c + ib)\, z^\mu \left(1 - z\right)^{\nu+1} F\left(\alpha + 1, \beta + 1; \gamma; z\right) \end{pmatrix},$$

$$z = -e^{-2i\varphi}\,,\ \mu = \frac{1}{2\omega}\sqrt{a^2 + (c - ib)^2}\,,\ \alpha = \mu + \nu + \lambda\,,$$

$$\nu = \frac{ib}{\omega}\,,\ \beta = \mu + \nu - \lambda\,,\ \gamma = 1 + 2\mu\,,\ \lambda = \frac{1}{2\omega}\sqrt{a^2 + (c + ib)^2}.$$

22. $F_1 = a$, $F_3 = b\tanh\varphi + c\coth\varphi$:

$$u = \begin{pmatrix} (2c + i\omega)z^\mu \left(1 - z\right)^\nu F\left(\alpha, \beta; \gamma; z\right) \\ az^{\mu+1/2} \left(1 - z\right)^\nu F\left(\alpha, \beta + 1; \gamma + 1; z\right) \end{pmatrix},$$

$$z = \tanh^2 \varphi\,,\ \mu = -\frac{ic}{2\omega}\,,\ \nu = \frac{i}{2\omega}\sqrt{a^2 + (b + c)^2}\,,$$

$$\gamma = \frac{1}{2} + 2\mu\,,\ \alpha = \gamma + \nu + \frac{i}{2\omega}\left(b + c\right)\,,\ \beta = \nu - \frac{i}{2\omega}\left(b + c\right)\,.$$

23. $F_1 = a$, $F_3 = (b \cosh 2\varphi + c) / \sinh 2\varphi$:

$$u = \left(\begin{array}{c} (b + c + i\omega)z^{\mu} (1 - z)^{\nu} F(\alpha, \beta; \gamma; z) \\ az^{\mu+1/2} (1 - z)^{\nu} F(\alpha, \beta + 1; \gamma + 1; z) \end{array} \right),$$

$$z = \tanh^2 \varphi , \; \mu = -\frac{i(b + c)}{4\omega} , \; \nu = \frac{i}{2\omega}\sqrt{a^2 + b^2} ,$$

$$\alpha = \frac{1}{2} + \nu - \frac{ic}{2\omega} , \; \beta = \nu - \frac{ib}{2\omega} , \; \gamma = \frac{1}{2} + 2\mu .$$

24. $F_1 = a$, $F_3 = (b \sinh \varphi + c) / \cosh \varphi$:

$$u = \left(\begin{array}{c} (2b + 2ic + i\omega)z^{\mu} (1 - z)^{\nu} F(\alpha, \beta; \gamma; z) \\ 2az^{\mu+1/2} (1 - z)^{\nu} F(\alpha, \beta + 1; \gamma + 1; z) \end{array} \right),$$

$$z = \left(\frac{e^{\varphi} + i}{e^{\varphi} - i} \right)^2 , \; \mu = \frac{c - ib}{2\omega} , \; \nu = \frac{i}{\omega}\sqrt{a^2 + b^2} ,$$

$$\alpha = \frac{1}{2} + \nu + \frac{c}{\omega} , \; \beta = \nu - \frac{ib}{\omega} , \; \gamma = \frac{1}{2} + 2\mu .$$

25. $F_1 = a$, $F_3 = b \tanh \varphi + c$:

$$u = \left(\begin{array}{c} az^{\mu} (1 - z)^{\nu} F(\alpha + 1, \beta; \gamma; z) \\ - (i2\omega\mu + b + c) z^{\mu} (1 - z)^{\nu} F(\alpha, \beta + 1; \gamma; z) \end{array} \right),$$

$$z = \frac{1}{2} (1 - \tanh \varphi) , \alpha = \mu + \nu + \frac{ib}{\omega} , \; \nu = \frac{i}{2\omega}\sqrt{a^2 + (b - c)^2} ,$$

$$\beta = \mu + \nu - \frac{ib}{\omega} , \; \gamma = 1 + 2\mu , \; \mu = \frac{i}{2\omega}\sqrt{a^2 + (b + c)^2} .$$

26. $F_1 = a$, $F_3 = b \coth \varphi + c$:

$$u = \left(\begin{array}{c} 2 (2b + i\omega) z^{\mu} (1 - z)^{\nu} F(\alpha, \beta; \gamma; z) \\ az^{\mu+1} (1 - z)^{\nu} F(\alpha + 1, \beta + 1; \gamma + 2; z) \end{array} \right),$$

$$z = 1 - e^{-2\varphi} , \mu = -\frac{ib}{\omega} , \; \nu = \frac{i}{2\omega}\sqrt{a^2 + (b + c)^2} , \; \alpha = \nu - \frac{ib}{\omega} + \lambda ,$$

$$\beta = \nu - \frac{ib}{\omega} - \lambda , \; \gamma = -\frac{2ib}{\omega} , \; \lambda = \frac{i}{2\omega}\sqrt{a^2 + (b - c)^2} .$$

9 Darboux transformation for the spin equation

Consider the SE with the external field

$$\mathbf{F}_{\varepsilon} = (F_1, 0, F_3) , \; F_1 = \varepsilon = \text{const}, \; F_3 = F_3(t) . \qquad (105)$$

Exact solutions for such external fields are presented in items 16–26 of the previous section. The SE with external fields of this form appears in various physical problems [14, 9, 10]. We consider here the Darboux transformation

[15] for the SE with the potentials (105). Such a transformation allows one to generate new exact solution from the known ones.

For the external field (105), the SE for the spinor V_ε can be written as the eigenvalue problem

$$\hat{h}V_\varepsilon = \varepsilon V_\varepsilon \,, \ \hat{h} = i\sigma_1\frac{d}{dt} + \Lambda \,, \ \Lambda = i\sigma_2 F_3 \,. \tag{106}$$

The idea of the Darboux transformation in this case can be formulated as follows: Suppose that the spinor V_ε is known for a given function F_3 with any complex ε. If an operator $\hat{L}$ (called the intertwining operator), that obeys the equation

$$\hat{L}\hat{h} = \hat{h}'\hat{L} \,, \hat{h}' = i\sigma_1\frac{d}{dt} + \Lambda' \,, \ \Lambda' = i\sigma_2 F_3' \tag{107}$$

for a function $F_3'(t)$ is known, then the eigenvalue problem

$$\hat{h}'V_\varepsilon' = \varepsilon V_\varepsilon' \tag{108}$$

can be solved as

$$V_\varepsilon' = \hat{L}V_\varepsilon \,. \tag{109}$$

If the intertwining operator $\hat{L}$ is chosen as

$$\hat{L} = \frac{d}{dt} + A \,, \tag{110}$$

where $A(t)$ is a time-dependent $n \times n$ matrix, then the transformation from V_ε to V_ε' is called the Darboux transformation [16]. There exists a general method of constructing the intertwining operators $\hat{L}$ (see, for example, [17] and references therein) for the given eigenvalue problem (106). However, for our purposes the direct application of the general method cannot be useful. The point is that applying this method one may violate the specific structure of the initial matrix Λ, so that the final matrix Λ' will not have the specific structure (107) with a real function F_3, whereas we wish to maintain the structure (107) of the matrix Λ', i.e., the structure (105) of the external field. Thus, the peculiarity of our problem is that the matrices Λ and Λ' must have the same block structure and the Darboux transformation must respect these restrictions. The existence of such transformations is a nontrivial fact, which we are going to verify below.

The intertwining relation (107) with the operator $\hat{L}$ in the form (110) and the matrix Λ' in the form (107) leads to the following relations:

$$\sigma_1 A - A\sigma_1 + \sigma_2 (F_3' - F_3) = 0 \,, \tag{111}$$

$$\sigma_1 \dot{A} + \sigma_2 A F_3' - \sigma_2 \dot{F}_3 - A\sigma_2 F_3 = 0 \,. \tag{112}$$

Let us choose

$$A = \alpha + i(F_3 - \beta)\sigma_3 \,, \tag{113}$$

where $\alpha(t)$ and $\beta(t)$ are certain functions. Then we obtain for the function F_3'

$$F_3' = 2\beta - F_3 \tag{114}$$

and the equations

$$\dot{\alpha} - 2\beta\left(F_3 - \beta\right) = 0\,, \ \dot{\beta} + 2\alpha\left(F_3 - \beta\right) = 0 \tag{115}$$

for the functions α and β. It is easy to see that there exists a first integral of equation (115):

$$\alpha^2 + \beta^2 = R^2\,, \ R = \mathrm{const}\,, \tag{116}$$

where R is a complex constant in the general case. Note that (116) is satisfied if we choose

$$\alpha = R\cos\mu\,, \ \beta = R\sin\mu\,, \tag{117}$$

with $\mu\left(t\right)$ being a real function. Substituting (117) into (115), we obtain for the function μ a nonlinear differential equation:

$$\dot{\mu} = 2\left(R\sin\mu - F_3\right)\,. \tag{118}$$

The time derivative in (110) can be taken from equation (106). Then we obtain, with allowance for (113) and (114),

$$V'_\varepsilon = \left[\alpha - i\left(\varepsilon\sigma_1 + \beta\sigma_3\right)\right]V_\varepsilon\,. \tag{119}$$

Thus, we can see that for the SE with the external field (105) there exists a Darboux transformation that does not violate the structure of the external field. It has the algebraic form (119) and is determined by solutions of equations (115), or by equations (118). To complete the construction, one has to represent solutions of the set (115), or (118), with the help of the initial solutions. Such a possibility does exist and is described below.

Let us fix $\varepsilon = \varepsilon_0$ and construct the vector

$$\mathbf{L} = \left(\bar{V}_{\varepsilon_0}, \sigma V_{\varepsilon_0}\right)\,, \tag{120}$$

see (135) from the Appendix. According to equation (33), this vector obeys the equation

$$\dot{\mathbf{L}} = 2\left[\mathbf{F}_{\varepsilon_0} \times \mathbf{L}\right]\,, \ \mathbf{F}_{\varepsilon_0} = \left(\varepsilon_0, 0, F_3\right)\,. \tag{121}$$

In addition, equations (136) and (130) from Appendix imply that

$$\mathbf{L}^2 = 0\,. \tag{122}$$

We construct solutions α and β of the set (115), via the Cartesian components $L_i\,, \ i = 1, 2, 3$, of $\mathbf{L}$, as follows:

$$\alpha = -\varepsilon_0\frac{L_2}{L_3}\,, \ \beta = -\varepsilon_0\frac{L_1}{L_3}\,. \tag{123}$$

Equation (122) implies

$$\alpha^2 + \beta^2 = -\varepsilon_0^2\,. \tag{124}$$

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Thus, we have expressed solutions of the set (115) via solutions of the initial equations (106) at $\varepsilon_0 = iR$. Substituting (123) into (119), one can find the final form of the Darboux transformation:

$$V'_\varepsilon = N\sigma_2 \left[(\sigma\mathbf{L}) L_3^{-1} + \frac{\varepsilon}{\varepsilon_0}\sigma_3 \right] V_\varepsilon \,, \tag{125}$$

where N is an arbitrary complex number. Using equation (147), (159) from the Appendix, we can transform (125) to a different form:

$$V'_\varepsilon = N \left[2\left(\bar{V}_\varepsilon, V_\varepsilon\right) L_3^{-1}\sigma_2 V_{\varepsilon_0} + i\frac{\varepsilon}{\varepsilon_0}\sigma_1 V_\varepsilon \right] \,. \tag{126}$$

For the constructed Darboux transformation, one can check the following properties:

1. If F_3 is real, then the function F'_3 is also real in case one chooses the functions α and β to be real. This choice is always possible, according to equations (115).

2. If F_3 is imaginary, then the function F'_3 is also imaginary in case one chooses α to be real and β to be imaginary. This choice is always possible, according to equations (115).

Consider, finally, a simple example of Darboux transformation. Let $F_3 = f$ be a constant. Then the general solution of the SE can be obtained from (101)–(103). It has the form

$$V_\varepsilon(t) = \left(\begin{array}{c} i\left(f - \omega\right) p \exp\left(i\omega t\right) - \varepsilon q \exp\left(-i\omega t\right) \\ i\varepsilon p \exp\left(i\omega t\right) + \left(f - \omega\right) q \exp\left(-i\omega t\right) \end{array} \right) \,, \quad \omega^2 = f^2 + \varepsilon^2 \,, \tag{127}$$

where p and q are arbitrary complex constants. The functions α and β can be easily found:

$$\alpha = -\frac{\dot{Q}}{2\left(Q - f\right)} \,, \quad \beta = f + \frac{f^2 - R^2}{Q - f} \,,$$
$$Q = R\cosh\varphi \,, \quad \varphi = 2\left(\omega_0 t + \varphi_0\right) \,, \quad \omega_0^2 = R^2 - f^2 \,, \tag{128}$$

where R and φ_0 are arbitrary complex constants. Then the function F'_3 can be found from (114):

$$F'_3 = f + 2\frac{f^2 - R^2}{Q - f} \,. \tag{129}$$

If f is real, then F'_3 is also real in case we choose a real R and a real φ_0 for $R^2 > f_0^2$. For $R^2 < f^2$, replacing φ_0 by $i\varphi_0$, we also obtain a real F'_3, which is determined by (129) with $Q = R\cos\varphi$ and $\omega_0 = \sqrt{|R^2 - f^2|}$. If f is imaginary, then F_3 is also imaginary in case we choose an imaginary R .

New solutions V'_ε of the SE can be easily constructed according to formula (119). We do not present here their explicit form, which is quite cumbersome. Note that for any $f \neq 0$ such solutions do not coincide with the new solutions presented in the previous section.

10 Appendix

10.1 Spinors and related vectors

The following relations hold for any spinor V and its anticonjugate spinor $\bar{V}$ (11):

$$\overline{\left(\bar{V}\right)} = -V \,, \; \left(\bar{V}, \bar{V}\right) = (V, V) \,, \; \left(\bar{V}, V\right) = \left(V, \bar{V}\right) = 0 \,. \tag{130}$$

Since $V \neq 0$ and $\bar{V}$ are orthogonal, they are linearly independent. Therefore, any spinor U can be represented as

$$U = (V, V)^{-1} \left[(V, U) V + \left(\bar{V}, U\right) \bar{V} \right] \,. \tag{131}$$

In fact, this means that the completeness relation

$$V V^+ + \bar{V} \, \bar{V}^+ = (V, V) I \tag{132}$$

takes place. We note that

$$V U^+ = \begin{pmatrix} v_1 u_1^* & v_1 u_2^* \\ v_2 u_1^* & v_2 u_2^* \end{pmatrix}, \; \det\left(V U^+\right) = 0 \,. \tag{133}$$

For any two spinors U and V, the following relations hold:

$$(U, \bar{V}) = -(V, \bar{U}), \; (\bar{U}, V) = -(\bar{V}, U), \; (\bar{U}, \bar{V}) = (V, U) \,,$$
$$(U, \bar{V})(\bar{V}, U) = (U, U)(V, V) - (U, V)(V, U) = (\bar{U}, V)(V, \bar{U}) \geq 0 \,. \tag{134}$$

For any two spinors U and V, we define a complex vector $\mathbf{L}^{u,v}$ as follows:

$$\mathbf{L}^{u,v} = (U, \sigma V) = \left(u_1^* v_2 + u_2^* v_1, \; i u_2^* v_1 - i u_1^* v_2, \; u_1^* v_1 - u_2^* v_2 \right) \,. \tag{135}$$

The following properties hold:

i) $\left(\mathbf{L}^{u,v}\right)^* = \mathbf{L}^{v,u} \,, \; \mathbf{L}^{\bar{u}, \bar{v}} = -\mathbf{L}^{v,u} \,,$

ii) $\mathbf{L}^{u,v} \mathbf{L}^{u',v'} = 2 (U, V') (U', V) - (U, V) (U', V') \,,$

iii) $\mathbf{L}^{v,v} \mathbf{L}^{v,v} = (V, V)^2 \,, \; \mathbf{L}^{\bar{v},v} \mathbf{L}^{\bar{v},v} = \mathbf{L}^{v,\bar{v}} \mathbf{L}^{v,\bar{v}} = 0 \,,$

iv) $\mathbf{L}^{\bar{v},v} \mathbf{L}^{v,\bar{v}} = 2 (V, V)^2 \,, \; \mathbf{L}^{v,v} \mathbf{L}^{\bar{v},v} = \mathbf{L}^{v,v} \mathbf{L}^{v,\bar{v}} = 0 \,,$

v) $\left[\mathbf{L}^{v,\bar{v}} \times \mathbf{L}^{\bar{v},v}\right] = 2i (V, V) \mathbf{L}^{v,v} \,, \; \left[\mathbf{L}^{\bar{v},v} \times \mathbf{L}^{v,v}\right] = i (V, V) \mathbf{L}^{\bar{v},v} \,,$

$\left[\mathbf{L}^{v,v} \times \mathbf{L}^{v,\bar{v}}\right] = i (V, V) \mathbf{L}^{v,\bar{v}} \,,$

vii) $\mathbf{L}^{u,v} = (V, V)^{-1} \left[(U, V) \mathbf{L}^{v,v} + \left(U, \bar{V}\right) \mathbf{L}^{\bar{v},v} \right] \,. \tag{136}$

For any vector $\mathbf{p}$ and any spinor V, we derive, with the help of (131),

$$(\sigma \mathbf{p}) V = (V, V)^{-1} \left[\left(\mathbf{L}^{v,v} \mathbf{p}\right) V + \left(\mathbf{L}^{\bar{v},v} \mathbf{p}\right) \bar{V} \right] \,. \tag{137}$$

Relations (136) imply that any spinor V produces three linearly independent vectors:

$$\mathbf{L}^{v,v}, \; \mathbf{L}^{\bar{v},v}, \; \mathbf{L}^{v,\bar{v}} \,. \tag{138}$$

Any complex vector $\mathbf{a}$ can be decomposed in these vectors:

$$\mathbf{a} = a_1 \mathbf{L}^{v,v} + a_2 \mathbf{L}^{\bar{v},v} + a_3 \mathbf{L}^{v,\bar{v}} \,,$$

$$a_1 = \frac{\mathbf{a}\mathbf{L}^{v,v}}{(V,V)^2} \,, \quad a_2 = \frac{\mathbf{a}\mathbf{L}^{v,\bar{v}}}{2\,(V,V)^2} \,, \quad a_3 = \frac{\mathbf{a}\mathbf{L}^{\bar{v},v}}{2\,(V,V)^2} \,. \tag{139}$$

It is useful to define three real orthogonal unit vectors $\mathbf{e}_i$, $i = 1,2,3$, with the help of the vectors (138),

$$\mathbf{e}_1 = \frac{\mathbf{L}^{v,\bar{v}} + \mathbf{L}^{\bar{v},v}}{2\,(V,V)} \,, \quad \mathbf{e}_2 = i\frac{\mathbf{L}^{v,\bar{v}} - \mathbf{L}^{\bar{v},v}}{2\,(V,V)} \,, \quad \mathbf{n} = \frac{\mathbf{L}^{v,v}}{(V,V)} \,. \tag{140}$$

The latter vectors obey the relations

$$\mathbf{e}_i \mathbf{e}_j = \delta_{ij} \,, \quad [\mathbf{e}_i \times \mathbf{e}_j] = \epsilon_{ijk} \mathbf{e}_k \,, \quad \mathbf{e}_3 = \mathbf{n} \,, \tag{141}$$

where ϵ_{ijk} is the Levi-Civita symbol ($\epsilon_{123} = 1$). The inverse relations have the form

$$\mathbf{L}^{v,v} = (V,V)\,\mathbf{n} \,, \quad \mathbf{L}^{\bar{v},v} = (V,V)\,(\mathbf{e}_1 + i\mathbf{e}_2) \,, \quad \mathbf{L}^{v,\bar{v}} = (V,V)\,(\mathbf{e}_1 - i\mathbf{e}_2) \,. \tag{142}$$

Any spinor V can always be represented as

$$V = Ne^{i\frac{\alpha}{2}} \left(\begin{array}{c} e^{-i\frac{\varphi}{2}} \cos\frac{\theta}{2} \\ e^{i\frac{\varphi}{2}} \sin\frac{\theta}{2} \end{array} \right) \,, \quad N^2 = (V,V) \,, \tag{143}$$

where N, α, θ, and φ are real numbers. The anticonjugate spinor reads

$$\bar{V} = Ne^{-i\frac{\alpha}{2}} \left(\begin{array}{c} -e^{-i\frac{\varphi}{2}} \sin\frac{\theta}{2} \\ e^{i\frac{\varphi}{2}} \cos\frac{\theta}{2} \end{array} \right) \,. \tag{144}$$

Considering θ and φ to be the angles of a spherical reference frame, we define the corresponding unit orthogonal vectors $\mathbf{e}_\varphi, \mathbf{e}_\theta$, and $\mathbf{n}$,

$$\begin{aligned}
\mathbf{e}_\theta &= (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) = [\mathbf{e}_\varphi \times \mathbf{n}] \,, \\
\mathbf{e}_\varphi &= (\sin\varphi, \cos\varphi, 0) = [\mathbf{n} \times \mathbf{e}_\theta] \,, \\
\mathbf{n} &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = [\mathbf{e}_\theta \times \mathbf{e}_\varphi] \,.
\end{aligned} \tag{145}$$

In terms of these vectors, the vectors (138) and (140) can be written as

$$\begin{aligned}
&\mathbf{L}^{v,v} = N^2\mathbf{n}, \quad \mathbf{L}^{\bar{v},v} = N^2\,(\mathbf{e}_\theta + i\mathbf{e}_\varphi)\,e^{i\alpha}, \quad \mathbf{L}^{v,\bar{v}} = N^2\,(\mathbf{e}_\theta - i\mathbf{e}_\varphi)\,e^{-i\alpha}, \\
&\mathbf{e}_1 = \mathbf{e}_\theta\cos\alpha - \mathbf{e}_\varphi\sin\alpha \,, \quad \mathbf{e}_2 = \mathbf{e}_\theta\sin\alpha + \mathbf{e}_\varphi\cos\alpha \,.
\end{aligned} \tag{146}$$

In addition, it follows from (137) and (146) that

$$\sigma\mathbf{F}V = \mathbf{n}\mathbf{F}V + (\mathbf{F}\mathbf{e}_\theta + i\mathbf{F}\mathbf{e}_\varphi)\exp(i\alpha)\,\bar{V} \,. \tag{147}$$

10.2 Eigenvalue problem

Let us consider the general eigenvalue problem in the space of two-dimensional spinors:

$$AV = \Lambda V \,. \tag{148}$$

Here, A is a given complex 2×2 matrix and Λ is an eigenvalue. For nontrivial solutions, the condition

$$\det(A - \Lambda I) = 0 \tag{149}$$

must hold. One can easily see that

$$\det(A - \Lambda I) = \Lambda^2 - \Lambda \operatorname{tr}A + \det A \,. \tag{150}$$

With allowance for (150), equation (149) implies that two nonzero eigenvalues are possible:

$$\Lambda_\zeta = \frac{1}{2}\left[\operatorname{tr}A + \zeta\sqrt{(\operatorname{tr}A)^2 - 4\det A}\right] \,, \ \zeta = \pm 1 \,. \tag{151}$$

On the other hand, any 2×2 matrix A can be written as

$$A = a_0 I + (\sigma\mathbf{a}) = \frac{1}{2}\operatorname{tr}A + (\sigma\mathbf{a}) \,. \tag{152}$$

Then the eigenvalue problem (148) can be written in the form

$$(\sigma\mathbf{a})V = \lambda V \,, \tag{153}$$

where $\lambda = \Lambda - \frac{1}{2}\operatorname{tr}A$.

For $\mathbf{a}^2 \neq 0$, there exist two nontrivial solutions (since $\det(\sigma\mathbf{a}) = -\mathbf{a}^2$). Multiplying (153) by $(\sigma\mathbf{a})$ from the left, and taking into account that $(\sigma\mathbf{a})(\sigma\mathbf{a}) = \mathbf{a}^2$, we obtain

$$\mathbf{a}^2 V = \lambda^2 V \implies \lambda_\zeta = \zeta\sqrt{\mathbf{a}^2}, \ \zeta = \pm 1 \,, \ \mathbf{a}^2 \neq 0 \,. \tag{154}$$

The corresponding eigenvectors V_ζ can be written as

$$V_1 = N_1 \begin{pmatrix} a_3 + \sqrt{\mathbf{a}^2} \\ a_1 + ia_2 \end{pmatrix}, \ V_{-1} = N_{-1} \begin{pmatrix} ia_2 - a_1 \\ a_3 + \sqrt{\mathbf{a}^2} \end{pmatrix}, \tag{155}$$

where N_ζ are normalization factors.

For $\mathbf{a}^2 = 0$, the matrix $(\sigma\mathbf{a})$ is singular, and there exists a nontrivial solution V_0 of equation (153) for $\lambda = 0$. Such a solution reads

$$V_0 = N \begin{pmatrix} ia_2 - a_1 \\ a_3 \end{pmatrix}, \ \mathbf{a}^2 = 0 \,. \tag{156}$$

In a sense, the inverse eigenvalue problem can be formulated as follows: For any two linearly independent spinors U and V, there exists a vector $\mathbf{a}$ such that

the eigenvalue problem (153) with this vector admits solutions V_ζ, $\zeta = \pm 1$, of the form

$$V_1 = U , \ V_{-1} = V . \tag{157}$$

The vector $\mathbf{a}$ is determined by the spinors U and V with accuracy up to a multiplier N,

$$\mathbf{a} = N\mathbf{L}^{\overline{u},v} , \tag{158}$$

where the vector $\mathbf{L}^{\overline{u},v}$ is defined by (135). To prove the above statement, we consider the matrix $A^{u,v}$,

$$A^{u,v} = \sigma \mathbf{L}^{u,v} = 2VU^+ - (U,V) I$$
$$= \begin{pmatrix} u_1^* v_1 - u_2^* v_2 & 2u_2^* v_1 \\ 2u_1^* v_2 & u_2^* v_2 - u_1^* v_1 \end{pmatrix} . \tag{159}$$

This matrix has the following properties:

$$A^{u,v} = (A^{v,u})^+ , \ \det A^{u,v} = -(U,V)^2 ,$$
$$A^{u,v}V = (U,V) V , \ A^{u,v}\bar{U} = -(U,V)\bar{U} ,$$
$$U^+ A^{u,v} = (U,V) U^+ , \ \bar{V}^+ A^{u,v} = -(U,V)\bar{V}^+ . \tag{160}$$

Then the proof of the statement follows from (159) and (160).

If we take the spinors U and V to be orthogonal,

$$(U,V) = 0 \Longleftrightarrow U = \alpha\bar{V} , \ \alpha = \text{const} , \tag{161}$$

then (158) is reduced to

$$\mathbf{a} = N'\mathbf{L}^{v,v}, \ N' = -\alpha^* N , \tag{162}$$

see (130). Note that the vector $\mathbf{L}^{v,v}$ is real.

In particular, the above consideration allows us to conclude that the eigenvectors U and V of equation (153) are orthogonal iff the vector $\mathbf{a}$ is a product of a complex factor and a real vector. In such a case, the spinors V and $\bar{V}$ obey the completeness relation (132).

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